Toward a Theoretical Foundation of Policy Optimization for Learning Control Policies

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L4DC Tutorial, Philadelphia, PA

June 14, 2023
Motivation

Data-guided decision-making for complex tasks in dynamical systems, e.g., game playing, robotics, networked systems,...

Many recent successes via Reinforcement Learning
Motivation: Policy Optimization

- A workhorse of (deep) RL: (direct) policy optimization methods
- Robotic manipulation, locomotion, video games, ChatGPT, etc.
Why is policy optimization popular?

- Easy-to-implement & scalable to high-dimensional problems
- Enable model-free search for complex dynamics (e.g. with rich contact) or rich observations (e.g. images)
- This tutorial: Does policy optimization have guarantees on linear control benchmarks (e.g. LQR, LQG, $\mathcal{H}_\infty$ control, etc)?
What is Policy Optimization (PO)?

- PO is an old idea from control: Fix the controller structure, and optimize a control metric over the parameters of the controllers

$$\min_K J(K)$$

- Parametrized controller/policy $K$
- Cost function $J$ (tracking errors, closed-loop $\mathcal{H}_2/\mathcal{H}_\infty$ norms, etc)

- Policy gradient method: $K' = K - \alpha \nabla J(K)$
- Example: Optimization-based PID Tuning $K = [K_p, K_i, K_d]^\top \in \mathbb{R}^3$

Credit: Astrom & Murray, 2020
History : Convex LMIs vs. PO

Key points :

- In 1980s, convex optimization methods become dominant due to strong global guarantees and efficient interior point methods
- PO problem formulation is generally not convex
- Reparameterize as convex optimization problems (one does not optimize the controller parameters directly); Lyapunov theory, stability/performance certificates, HJB, ...
- Examples of LMIs: state-feedback or full-order output-feedback $\mathcal{H}_2/\mathcal{H}_\infty$ control
- e.g., Boyd et al., “Linear Matrix Inequalities in System and Control Theory”, 1994, SIAM
Historically, PO is used for control problems that can’t be convexified; often no theory

- Sometimes the plant order is unknown: PID tuning, feedback iterative tuning, etc
- Static output feedback LQ control
- Fixed-order structured $\mathcal{H}_\infty$ synthesis: HIFOO and Hinfstruct [Apkarian and Dominikus, ’06]
- Distributed control design: Martensson/Rantzer (’09)

In recent years, new reason to revisit PO for classical control: help provide theory towards understanding model-free RL
A Modern Perspective from Deep RL

A common practice nowadays in deep RL for robotic control: **visuomotor policy learning/image-to-torque** [Levine et al., ’16]

- A type of **perception-based control**: purely **model-free**
- Train **perception** and **control** systems jointly **end-to-end**

Advantages:

- Direct and relatively simple to implement
- Mitigate **compounding error** as in model-based RL (separately train perception and control)
- Make better use of deep NNs’ abstraction and perception capabilities to handle **high-dimensional visual signals**
Vanilla policy gradient:

- Policy Gradient Theorem [Sutton et al., '99]
  \[ \nabla J(K) = \mathbb{E}[Q_K(x, u) \cdot \nabla \log \pi_K(u \mid x)] \n\]

- REINFORCE estimator [Williams '92]: from \( N \) trajectories of length \( T \) – \((x_{t,i}, u_{t,i}, c_{t,i})_{i \in [N], t \in [T]} \)
  \[ \nabla J(K) \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \left[ \left( \sum_{\tau=t}^{T} c_{\tau,i} \right) \cdot \nabla \log \pi_K(u_{t,i} \mid x_{t,i}) \right] \]

- Others estimators: \( \text{G(PO)} \)MDP [Baxter & Bartlett, '01], actor-critic [Konda & Tsitsiklis, '99], natural policy gradient [Kakade '01] (will come back to it!)

- Essentially stochastic gradient descent (SGD) (heart of modern machine learning)!

Modern variants (benefit from the advances of optimization theory):

- Deep deterministic PG (DDPG) [Silver et al., '14], Trust-region PO (TRPO) [Schulman et al., '15], Proximal PO (PPO) [Schulman et al., '17], soft actor-critic (SAC) [Haarnoja et al., '18], variance-reduced PG [Papini et al., '18]...

- Default algorithm in OpenAI Gym, Dota 5v5, ChatGPT training – PPO
Missing Perspectives in Deep RL Literature

- Convergence guarantees: Nonconvex optimization in policy parameter spaces, e.g., weights of neural networks
- Sample efficiency guarantees: How many samples are needed? Polynomial in problem parameters?
- Constraints: Stability and robustness of the closed-loop systems
Missing Perspectives in Classic Control Literature

- Landscape: Is convexity really needed for optimization?
- Finite-iteration/sample complexity: If an algorithm converges, how fast and how many samples are needed?

GD on convex landscape

GD for nonconvex landscape
Tutorial Overview
This tutorial: Understanding policy optimization via examining guarantees on linear control benchmarks

- Start from simpler contexts and gain insights
  - Classical control benchmarks (c.f. [Recht et al., ’17])
- Identify issues for establishing guarantees of PO for control
- Employ modern optimization perspective: iteration/sample complexity, first-order & zeroth-order oracle models, etc

Big picture:

- One perspective to bridge control theory and RL
- Understand and connect “model-free” & “model-based” views
- Towards a general framework for learning-based control
Schedule

- Now-2 :30pm : Preview and Some Optimization Background
- 2 :30-3 :00pm : PO Theory for LQR
- 3 :00-3 :30pm : PO Theory for Risk-sensitive & \( \mathcal{H}_2/\mathcal{H}_\infty \) Robust Control
- 3 :30-4 :00pm : Coffee Break
- 4 :00-4 :30pm : PO Theory for State-feedback \( \mathcal{H}_\infty \) Synthesis
- 4 :30-5 :00pm : PO Theory for LQG
- 5 :00-5 :15pm : Role of convex parameterization
- 5 :15-5 :30pm : Future work and Q&A/discussions
Revisit linear control problems as benchmarks for PO

\[
\min_K J(K), \quad s.t. \ K \in \mathcal{K}
\]

- Parametrized policy $K$ (e.g. linear mapping, neural networks)
- Cost function $J$ (tracking errors, closed-loop $H_2/H_\infty$ norms, etc)
- Constraint set $\mathcal{K}$ (stability, robustness, safety, etc)

Policy gradient: $K' = K - \alpha \nabla J(K)$

- The gradient $J$ can be estimated from data in a model-free manner (policy gradient theorem or stochastic finite difference)
- For nonsmooth problems, replace the gradient with some subgradient

Recent progress on PO theory (Nonconvexity is the key issue)

- Landscape: Is stationary global minimum?
- Feasibility: Does the policy search stay in the feasible set $\mathcal{K}$?
- Global convergence & sample complexity

Linear quadratic regulator (LQR) as PO: Consider \( x_{t+1} = Ax_t + Bu_t + w_t \)

\[
\min_K J(K) := \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} (x_t^TQx_t + u_t^TRu_t) \right], \quad \text{s.t. } K \text{ is stabilizing}
\]

- \( u_t = -Kx_t \) for gain matrix \( K \)
- \( \mathcal{K} = \{ K : \rho(A - BK) < 1 \} \); \( \mathcal{K} \) a nonconvex constraint set

PO theory for LQR

- Landscape: Feasible set is connected, and stationary is global
- Feasibility: The LQR cost is coercive and serves as a barrier on \( \mathcal{K} \)
- Global convergence & sample complexity: Linear rate and finite sample complexity via the gradient dominance/smoothness property

Main Ref:

Mixed design: $\mathcal{H}_\infty$ constraints are crucial for robustness

$$\min_K J(K), \quad \text{s.t. } K \text{ is stabilizing and robust in the } \mathcal{H}_\infty \text{ sense}$$

$J(K)$ is an upper bound on the $\mathcal{H}_2$ performance

$u_t = -Kx_t$ for gain matrix $K$

$\mathcal{K} = \{ K : \rho(A - BK) < 1; \| T(K) \|_\infty < \gamma \} ;$ add robustness constraints

$\gamma \to \infty$ reduces to LQR

PO theory for mixed design

Key issue: The cost is not coercive! How to maintain feasibility?

Fix: Implicit regularization via Natural policy gradient (NPG) and Gauss-Newton

Global sublinear convergence for NPG and Gauss-Newton

Main Ref:

Preview: $\mathcal{H}_\infty$ State-Feedback Synthesis as PO

- $\mathcal{H}_\infty$ state-feedback synthesis: $x_{t+1} = Ax_t + Bu_t + w_t$ with $x_0 = 0$
  \[ \min_K J(K), \quad s.t. \quad K \text{ is stabilizing} \]
  \[ J = \sum_{t=0}^{\infty} (x_t^T Q x_t + u_t^T R u_t) \] subject to the worst-case disturbance satisfying $\sum_{t=0}^{\infty} \| w_t \|^2 \leq 1$
  \[ u_t = -Kx_t \text{ for gain matrix } K \]
  \[ K = \{ K : \rho(A - BK) < 1 \} \]
  \[ J(K) \text{ is the closed-loop } \mathcal{H}_\infty \text{ norm (nonsmooth in } K)! \]

- PO theory for $\mathcal{H}_\infty$ state-feedback synthesis (Nonconvex and nonsmooth)
  - Key issue: The cost may not be differentiable at important points
  - Fix: Show that Clarke stationary points are global, and apply Goldstein’s subgradient method to guarantee sufficient descent
  - Global convergence: Goldstein’s subgradient method achieves global convergence provably

- Main Ref:
Linear quadratic Gaussian (LQG) is the partially observable variant of LQR, and can be treated as PO (more details later).

PO theory for LQG

- Issue 1: Feasible set may not be connected
- Issue 2: Stationary may not be global
- Today’s talk: Some positive results and many open questions

Main Ref:

Background: Optimization Theory
**Optimization of Smooth Nonconvex Functions**

**Definition**: A function $J(K)$ is $L$-smooth if the following inequality holds for all $(K, K')$:

$$J(K') \leq J(K) + \langle \nabla J(K), (K' - K) \rangle + \frac{L}{2} \| K' - K \|^2_F.$$

The above definition is equivalent to $\nabla J$ being $L$-Lipschitz.

**Complexity**: Gradient descent method $K^{n+1} = K^n - \alpha \nabla J(K^n)$ is guaranteed to find $\epsilon$-stationary point of $J$ within $O\left(\frac{1}{\epsilon^2}\right)$ steps

$$J(K^{n+1}) \leq J(K^n) + \langle \nabla J(K^n), K^{n+1} - K^n \rangle + \frac{L}{2} \| K^{n+1} - K^n \|^2_F$$

$$= J(K^n) + \left(-\alpha + \frac{L\alpha^2}{2}\right) \| \nabla J(K^n) \|^2_F,$$

Summing the above inequality from $n = 0$ to $T$

$$\left(\alpha - \frac{L\alpha^2}{2}\right) \sum_{n=0}^{T} \| \nabla J(K^n) \|^2_F \leq J(K^0) - J(K^{n+1})$$
Optimization of Smooth Nonconvex Functions

**Complexity**: Gradient descent method $K^{t+1} = K^n - \alpha \nabla J(K^n)$ is guaranteed to find $\epsilon$-stationary point of $J$ within $O\left(\frac{1}{\epsilon^2}\right)$ steps

$$
\left(\alpha - \frac{L\alpha^2}{2}\right) \sum_{n=0}^{T} \|\nabla J(K^n)\|_F^2 \leq J(K^0) - J(K^{n+1})
$$

If $\alpha < \frac{2}{L}$, then $C = \alpha - \frac{L\alpha^2}{2} > 0$. We know $J(K^{n+1}) \geq J^*$ for some $J^*$.

$$
\sum_{n=0}^{T} \|\nabla J(K^n)\|_F^2 \leq \frac{J(K^0) - J^*}{C}
$$

$$
\implies \min_{0 \leq n \leq T} \|\nabla J(K^n)\|_F^2 \leq \frac{1}{T + 1} \sum_{n=0}^{T} \|\nabla J(K^n)\|_F^2 \leq \frac{J(K^0) - J^*}{C(T + 1)}.
$$

To find a point whose gradient norm is smaller than or equal to $\epsilon$, we need to run $T$ steps with

$$
T = \frac{J(K^0) - J^*}{C\epsilon^2} - 1 = O\left(\frac{1}{\epsilon^2}\right).
$$

which is the complexity for finding $\epsilon$-approximate stationary point
Optimization of Smooth Nonconvex Functions

**Complexity**: Gradient descent method $K^{t+1} = K^n - \alpha \nabla J(K^n)$ is guaranteed to find $\epsilon$-stationary point of $J$ within $O\left(\frac{1}{\epsilon^2}\right)$ steps.

**Convergence**: Gradient descent method is guaranteed to convergence to a stationary point eventually.

**Question**: What if we can show stationary is global?

**Answer**: Then the gradient descent method converges to global minimum! We have $J(K^n) \to J^*$!

**Take-away**: Nonconvex optimization may not be that terrifying if stationary is global!
Gradient Dominance and Linear Rate to Global Minimum

**Definition:** A function $J(K)$ is gradient dominant if it is continuously differentiable and satisfies

$$J(K) - J(K^*) \leq \frac{1}{2\mu} \|\nabla J(K)\|_F^2,$$

$\forall K \in \mathcal{K},$

**Landscape:** Stationary is global!

**Complexity:** Gradient descent method $K^{n+1} = K^n - \alpha \nabla J(K^n)$ is guaranteed to find $\epsilon$-optimal point of $J$ within $O\left(\log \left(\frac{1}{\epsilon}\right)\right)$ steps

$$J(K^{n+1}) \leq J(K^n) + \left(-\alpha + \frac{L\alpha^2}{2}\right) \|\nabla J(K^n)\|_F^2,$$

$$\leq J(K^n) - 2\mu \left(\alpha - \frac{L\alpha^2}{2}\right) (J(K^n) - J^*)$$

$$\Rightarrow J(K^{n+1}) - J^* \leq (1 - 2\mu\alpha + \mu L\alpha^2)(J(K^n) - J^*)$$

$$\Rightarrow J(K^T) - J^* \leq (1 - 2\mu\alpha + \mu L\alpha^2)^T (J(K^0) - J^*)$$

Running $T$ steps with $T = O\left(\log \left(\frac{1}{\epsilon}\right)\right)$ guarantees $J(K^T) - J^* \leq \epsilon$
What if there are constraints? If the cost is coercive, then it is a barrier function by itself!

**Definition**: A function $J(K)$ is coercive on $\mathcal{K}$ if for any sequence $\{K^i\}_{i=1}^{\infty} \subset \mathcal{K}$ we have $J(K^i) \to +\infty$ if either $\|K^i\|_2 \to +\infty$, or $K^i$ converges to an element on the boundary $\partial \mathcal{K}$. 
A Useful Result for Constrained Optimization

If $J$ is coercive and twice continuously differentiable on $\mathcal{K}$, we have

▶ The sublevel set $\mathcal{K}_\gamma := \{ K \in \mathcal{K} : J(K) \leq \gamma \}$ is compact.

▶ The function $J(K)$ is $L$-smooth on $\mathcal{K}_\gamma$, and the constant $L$ depends on $\gamma$ and the problem parameters.

▶ Suppose running GD method $K^{n+1} = K^n - \alpha \nabla J(K^n)$ initialized from $K^0 \in \mathcal{K}$. Let $L$ be the smoothness parameter for $\mathcal{K}_J(K^0)$. Then GD finds an $\epsilon$-approximate stationary point with $O\left(\frac{1}{\epsilon^2}\right)$ steps with $\alpha = 1/L$.

▶ If $J$ is gradient dominant with parameter $\mu$, then GD achieves linear convergence rate.

$$J(K_T^T) - J^* \leq (1 - 2\mu \alpha + \mu L \alpha^2)^T (J(K^0) - J^*)$$
PO Theory for LQR
Linear quadratic theory – background

Standard LQR problem (discrete-time, infinite horizon): linear dynamics

\[ x_{t+1} = Ax_t + Bu_t \]

with given initial state \( x_0 \), choose control sequence

\[ u_0, u_1, \ldots, u_t, \ldots \]

in order to minimize the total cost

\[ \sum_{t=0}^{\infty} x_t^\top Q x_t + u_t^\top R u_t \]

with given cost matrices \( Q, R \succ 0 \).
Linear quadratic theory

Classical solution via dynamic programming (when $A, B$ known, stabilizable): solve the \textit{algebraic Riccati equation} (for $P$)

$$P = Q + A^T PA - (A^T PB)(R + B^T PB)^{-1}(B^T PA)$$

then let

$$u_t = -K^* x_t = -(R + B^T PB)^{-1}(B^T PA) x_t$$

▶ a “go-to” model-based control design (since Kalman in 60’s)
▶ extensive theory, computational methods for solving Riccati equation (Laub; Kleinman ‘68; Hewer ‘71)
Value and policy iterations

The solution of ARE determines the value matrix
\[ \min_u J(x_0, u) = x_0^\top P^* x_0 \]
one can develop an iteration on \( P \) s. t. \( P \rightarrow P^* \), then recover the optimal control policy (this would be called value iteration)

PO for LQR, on the other hand, would directly update \( K \), e.g.,
\[ K^{n+1} = K^n - \eta d_K \]
when \( d_K \) is some sort of gradient update and \( \eta \) is (possibly time-varying) stepsize; this is a first order method

*this tutorial : can we develop direct PO methods with guarantees for some typical control synthesis problems?*
Direct policy optimization

towards writing LQR as “$J(K)$” ...

First, note that when $A$ is Schur stable, the sequence

$$\sum_{t}(A^\top)^tQA^t \to P$$

converges, where

$$P = A^\top PA + Q$$

so with stabilizing feedback in place, the LQR cost for the dynamics,

$$x_{t+1} = (A - BK)x_t$$

with an initial condition $x_0$, can be written as $x_0^TP_Kx_0$, where

$$P_K = (A - BK)^\top P_K(A - BK) + K^\top RK + Q$$

so in this case, LQR is really optimizing

$$\min_{P,K} \text{trace } P \Sigma_0 \quad (\text{with } \Sigma_0 = x_0x_0^\top)$$

$$P = (A - BK)^\top P(A - BK) + K^\top RK + Q$$
However, as stated, this problem is a bilinear matrix optimization ...

When $K \in S$ (set of stabilizing $K$), equation

$$P = (A - BK)^\top P (A - BK) + K^\top RK + Q$$

has a unique solution $P(K)$; hence, the LQR can be written (for a given $\Sigma$) as

$$\min_{K \in S} J(K)$$

We take $x_0 \sim \mathcal{D}(0, \Sigma_0)$ where $\Sigma_0$ is a full-rank covariance (equivalently, can take $\Sigma$ to correspond to a spanning set of initial conditions) thus $J$ is real analytic function over its domain.
Consider now PO algorithms:

- iterate on policy $K$,
- using gradient of cost, $\nabla J(K)$ (exact or approximate)

- does GD (with exact gradients) converge? under what assumptions? does it converge to the global opt $K^*$?
- rate of convergence?
- how about related algorithms, e.g., “natural gradient” descent?
- “model-free” version: if gradients not available, would sampling $J(K)$ work? finite-sample complexity?

**note**: challenging as $J(K)$ is **not convex**
LQR and policy gradient methods

- Consider LQR without state noise (for simplicity), random initial condition $x_0 \sim \mathcal{D}$
- Let $J(K)$ be the cost as function of policy $K$
- Define covariance matrices:

$$\Sigma_K = \mathbb{E} \left[ \sum_{t=0}^{\infty} x_t x_t^T \right], \quad \Sigma_0 = \mathbb{E} \left[ x_0 x_0^T \right]$$

- Consider algorithms:

  Gradient descent: $K \leftarrow K - \eta \nabla J(K)$

  Natural GD: $K \leftarrow K - \eta \nabla J(K) \Sigma_K^{-1}$

  [Kakade '01]
Suppose algorithms have only **oracle access** to the model \((A,B)\) not known explicitly), e.g.,

- **exact gradient oracle**: \(\nabla J(K)\)
- “approximate gradient” oracle: use sample values of \(J(K)\)

i.e., first-order and zeroth order oracle in optimization
Optimization landscape I

\[
J(K) = \langle \Sigma_0, P_K \rangle \\
= \text{vec}(\Sigma_0)^T (I - (A - BK) \otimes (A - BK))^{-1} \text{vec}(Q + K^T R K)
\]

**observation**: \( J(K) \) is **not** convex in \( K \) (or quasiconvex, star convex) for \( n \geq 3 \) (convex for single input case when \( n = 2 \)); we can computed the \( \nabla J(K) = 2(RK - B^TP_KA_K)\Sigma_K \), where,

\[
\Sigma_K = \mathbb{E} \left[ \sum_{t=0}^{\infty} x_t x_t^T \right] ; \quad \Sigma_0 = \mathbb{E} \left[ x_0 x_0^T \right]
\]

**lemma**: if \( \nabla J(K) = 0 \), then either

- \( K \) is optimal, or
- covariance \( \Sigma_K \) is rank deficient.

if \( \Sigma_0 \) is full rank, then \( \Sigma_K \) is full rank \( \implies \) stationary points globally optimal

can also show this leveraging the convex LMI reformulation—later research has shown this deeper connection and its uses.)
\[ J(K) = \langle \Sigma_0, P_K \rangle \]
\[ = \text{vec}(\Sigma_0)^T (I - (A - BK) \otimes (A - BK))^{-1} \text{vec}(Q + K^T RK) \]

**Observation:** \( J(K) \) is **not** convex in \( K \) (or quasiconvex, star convex) for \( n \geq 3 \).

\[ \Sigma_K = \mathbb{E} \left[ \sum_{t=0}^{\infty} x_t x_t^T \right], \quad \Sigma_0 = \mathbb{E} \left[ x_0 x_0^T \right] \]

**Lemma:** if \( \nabla J(K) = 0 \), then either

- \( K \) is optimal, or
- covariance \( \Sigma_K \) is rank deficient.

**If** \( \Sigma_0 \) **is full rank**, then \( \Sigma_K \) is full rank \( \implies \) stationary points globally optimal.

**Can also examine this via transformation to convex LMI (but proofs no simpler).**
**Lemma**: Suppose $\Sigma_0$ is full rank, then

$$J(K) - J(K^*) \leq \frac{\|\Sigma K^*\|}{\sigma_{\text{min}}(\Sigma_0)\sigma_{\text{min}}(R)} \|\nabla J(K)\|^2.$$

i.e., $J(K)$ gradient-dominated ([Polyak '63],...
Main Theory: Summary

[Fazel, Ge, Kakade, Mesbahi 2018] for discrete-time LQR

- $J(K)$ is generally not convex/quasiconvex/star-convex
- convex combination of two stabilizing $K$’s may not stabilize
- coerciveness: LQR cost is coercive on the set of stabilizing policies
- LQR cost is gradient dominant
- hence, gradient descent converges to $K^*$ from any stabilizing initial $K_0$! (with a linear rate)
- similarly for related algorithms, e.g., natural policy gradients and policy iteration algorithm
**Theorem 1**: Suppose $J(K_0)$ is finite (i.e., $K_0$ is stabilizing), $\Sigma_0$ is full rank. With stepsize $\eta$ chosen appropriately, and $\#$ of iterations $N$ as

- for natural policy GD:
  \[
  N \geq \frac{\|\Sigma K_*\|}{\sigma_{\min}(\Sigma_0)} \left( \frac{\|R\|}{\sigma_{\min}(R)} + \frac{\|B\|^2 J(K_0)}{\sigma_{\min}(\Sigma_0) \sigma_{\min}(R)} \right) \log \frac{J(K_0) - J(K^*)}{\epsilon},
  \]

- for GD:
  \[
  N \geq \frac{\|\Sigma K_*\|}{\sigma_{\min}(\Sigma_0)} \log \frac{J(K_0) - J(K^*)}{\epsilon} \poly(\text{everything else}),
  \]

then,

$K_N$ has cost $\epsilon$-close to optimum.
Numerical experiment

Left: Gradient descent for continuous time LQR
Right: 2-dim projection of the LQ cost contour
Unknown model case

Gradient descent: \[ K \leftarrow K - \eta \nabla J(K) \]

Natural policy GD: \[ K \leftarrow K - \eta \nabla J(K) \Sigma^{-1}_K \]

- we do not know (or directly learn) \( A, B \)
  - but have the ability to explore by perturbing \( K \)
- model-free estimation: add Gaussian noise to actions during rollouts
- similar to zeroth order (derivative-free) optimization
- **issues**: how much noise? length of rollouts? overall sample complexity?
Algorithm

Input : $K$, # trajectories $m$, rollout length $\ell$, parameter $r$, dimension $d$

for $i = 1, \cdots m$,

- draw $U_i$ is uniformly at random from $\|U\|_F \leq r$
- sample policy $\hat{K}_i = K + U_i$
- simulate $\hat{K}_i$ for $\ell$ steps starting from $x_0 \sim \mathcal{D}$.
- get empirical estimates

\[ \hat{C}_i = \sum_{t=1}^\ell c_t, \quad \hat{\Sigma}_i = \sum_{t=1}^\ell x_t x_t^\top \]

where $c_t$, $x_t$ are costs and states on this trajectory

end for

use following estimates for PGD/NPGD:

\[ \nabla J(K) = \frac{1}{m} \sum_{i=1}^m \frac{d}{r^2} \hat{C}_i U_i, \quad \hat{\Sigma}_K = \frac{1}{m} \sum_{i=1}^m \hat{\Sigma}_i \]
Idea for the Proof of Convergence

1. Prove when rollout length $\ell$ is large enough, cost function $C$ and covariance $\Sigma$ are close to infinite horizon quantities
2. Show with enough samples, alg can estimate gradient and covariance matrix within the desired accuracy
3. Show GD and NPGD converge with a similar rate, despite bounded perturbations in gradient/natural gradient estimates
Suppose $J(K_0)$ is finite, $\mu > 0$, $x_0$ has norm bounded by $L$ almost surely; and GD and the NPGD are called with parameters:

$$m, \ell, 1/r = \text{poly} \left( J(K_0), \frac{1}{\mu} \sigma_{\min}(Q), \|A\|, \|B\|, \|R\|, \frac{1}{\sigma_{\min}(R)}, d, 1/\epsilon, L^2/\mu \right).$$

- **NPGD**: for stepsize $\eta = \frac{1}{\|R\| + \|B\|^2 J(K_0)/\mu}$ and

$$N \geq \frac{\|\Sigma_{K^*}\|}{\mu} \left( \frac{\|R\|}{\sigma_{\min}(R)} + \frac{\|B\|^2 J(K_0)}{\mu \sigma_{\min}(R)} \right) \log \frac{2(J(K_0) - J(K^*))}{\epsilon},$$

with high probability NPGD satisfies: $J(K_N) - J(K^*) \leq \epsilon$

- **GD**: for appropriate stepsize $\eta$,

$$\eta = \text{poly} \left( \frac{\mu \sigma_{\min}(Q)}{J(K_0)}, \|A\|, \|B\|, \|R\|, \sigma_{\min}(R) \right)$$

and

$$N \geq \frac{\|\Sigma_{K^*}\|}{\mu} \log \frac{J(K_0) - J(K^*)}{\epsilon} \text{poly} \left( \frac{J(K_0)}{\mu \sigma_{\min}(Q)}, \|A\|, \|B\|, \|R\|, \frac{1}{\sigma_{\min}(R)} \right),$$

with high probability, GD satisfies: $J(K_N) - J(K^*) \leq \epsilon$
Related Results

A burst of recent research interest:

**LQR, continuous-time**: [Mohammadi et al., 2019], [Bu et al., 2020]

**LQR, discrete-time**: [Fazel et al., 2018], [Bu et al., 2019]

**Stabilization**: [Perdomo et al., 2021]

**Decentralized finite-horizon LQR under QI**: [Furieri et al., 2020]

**LQ games**: [Zhang et al., 2019], [Bu et al., 2019], [Mazumdar et al., 2019], [Hambly et al., 2021]

**Markov jump linear systems**: [Jansch-Porto et al., 2020]

**Output estimation with differentiable convex liftings (DCL) framework**: [Umenberger et al., 2022]

**$\mathcal{H}_\infty$ control**: [Tang and Zheng, 2023]

**Fundamental limits of policy gradient**: [Ziemann et al., 2022]

And many other variants! (See our survey article)

Coming up …

- 3:00-3:30pm: PO Theory for Risk-sensitive & $\mathcal{H}_2/\mathcal{H}_\infty$ Robust Control
- 3:30-4:00pm: Coffee Break
- 4:00-4:30pm: PO Theory for State-feedback $\mathcal{H}_\infty$ Synthesis
- 4:30-5:00pm: PO Theory for LQG
- 5:00-5:15pm: Role of convex parameterization
- 5:15-5:30pm: Future work and Q&A/discussions
Policy Optimization for Risk-Sensitive Control, $\mathcal{H}_2/\mathcal{H}_\infty$ Robust Control, and Linear Quadratic Games

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5th Annual Learning for Dynamics & Control Conference (L4DC)

University of Pennsylvania, PA       June 14, 2023
General Background: Recall

- Reinforcement learning (RL) has achieved tremendous empirical successes, including in some continuous-space control tasks
  - Game of Go, video games, robotics, etc

- A resurgence of interest in the theoretical understandings of RL

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1source: google images
Motivation

- Most scenarios involve more than one agent, e.g., game of Go, video games, robot team motion-planning, autonomous driving
- Most control systems are safety-critical, e.g., robotics, unmanned (aerial) vehicles, cyber-physical systems

**Goal:** Provable RL with robustness and multi-agency considerations
Background
Theoretical Understanding of Policy Optimization

- One workhorse in RL: Direct policy search/policy optimization
- Whether, where, how fast, PO methods converge?
  - Nonconvex in policy parameter space
- Let’s start with benchmark RL/control tasks before deep RL?
- PO for linear quadratic regulator (LQR) (and variants) has been extensively studied recently [Fazel et al., ’18][Tu & Recht, ’19][Malik et al., ’19][Bu et al., ’19][Mohammadi et al., ’19][Gravell et al. ’19][Li et al., ’19][Furieri et al., ’19]...

All models are wrong, so is LQR $\implies$ robustness concern is critical

- Question: whether & how PO methods can address benchmark control/RL with robustness/risk-sensitivity concerns?
- Side motivation: multi-agent RL (will come back later)
PO for RL/Control $\implies$ Optimization
PO for RL/Control \implies (Constrained Nonconvex) Opt.

- PO for RL/Control can be generally written as
  \[
  \min_K \ J(K), \quad \text{s.t.} \ K \in \mathcal{K}
  \]
  - Parametrized policy/controller \( K \)
  - Objective to optimize \( J \)
  - Constraint set \( \mathcal{K} \) (important but sometimes \textit{implicit}!)

- Linear quadratic regulator (LQR) as an example
  \[
  \min_K \ J(K) := \sum_{t=0}^{\infty} \mathbb{E}[x_t^\top Q x_t + u_t^\top R u_t], \quad \text{s.t.} \ K \text{ is stabilizing}
  \]
  - \( u_t = -K x_t \) for gain matrix \( K \)
  - \( \mathcal{K} = \{ K \mid \rho(A - BK) < 1 \} \); \( \mathcal{K} \) a \textit{nonconvex} constraint set

- Other examples of \( \mathcal{K} \): boundedness of \( K \)'s norm, probability simplex, safety-constraint on states, etc.
Robustness Constraint on $K$

- Beyond stability, robustness is a core topic in control theory
- Need a controller robust to disturbance/model-uncertainty

\[
G - \Delta \text{ model covers many robustness considerations}
\]

- Parametric uncertainty $\Delta A, \Delta B$ in $A, B$ (most popular in recent ML for control literature)
- Time-varying parameters $A_t, B_t$
- Time-varying delay $u_t = -Kx_{t-\tau}$
- Even dynamical uncertainty (unknown model-order)

- Robustness constraint $\mathcal{K}$?
Questions

▶ How to enforce/maintain robustness for policy-optimization RL methods during learning?

▶ What are the global convergence guarantees, if any, of PO methods in learning for robust control?
Problem Statement
Starting Point: LEQG [Jacobson '73]

- LQR/LQG \(\iff\) linear exponential quadratic Gaussian (LEQG)
- Simple but benchmark risk-sensitive control setting
- Linear system dynamics:

\[
x_{t+1} = Ax_t + Bu_t + w_t,
\]

system state \(x_t \in \mathbb{R}^d\) with \(x_0 \sim \mathcal{N}(\mathbf{0}, X_0)\), noise \(w_t \sim \mathcal{N}(\mathbf{0}, W)\)

- One-stage cost \(c(x, u) = x^\top Qx + u^\top Ru\), with objective

\[
\min J := \lim_{T \to \infty} \frac{1}{T} \log \mathbb{E} \exp \left[ \frac{\beta}{2} \sum_{t=0}^{T-1} \left( x_t^\top Qx_t + u_t^\top Ru_t \right) \right] 
\]

- Intuition: by Taylor expansion around \(\beta = 0\)

\[
J \approx \lim_{T \to \infty} \frac{1}{T} \left\{ \mathbb{E} \left[ \sum_{t=0}^{T-1} c(x_t, u_t) \right] + \frac{\beta}{4} \text{Var} \left[ \sum_{t=0}^{T-1} c(x_t, u_t) \right] \right\} + O(\beta^2).
\]
Starting Point: LEQG

- Optimal controller is LTI state-feedback [ZHB, '21], conjectured in [Glover and Doyle, '88]

\[ \mu_t(x_{0:t}, u_{0:t-1}) = -K^* x_t \]

- See more results on LEQG specifically in [ZHB, '21]
- Implicit robustness constraint in LEQG

**Lemma (Glover and Doyle '88)**

The feasible set of \( J(K) \) is the \( 1/\sqrt{\beta} \)-sublevel set of the \( \mathcal{H}_\infty \)-norm of \( T(K) \), i.e.,

\[
\{ K \mid K \text{ stabilizing; } \| T(K) \|_\infty < 1/\sqrt{\beta} \}.
\]
Bigger Picture: Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control

- Linear dynamic systems:
  \[ x_{t+1} = Ax_t + Bu_t + Dw_t \quad \text{and} \quad z_t = Cx_t + Eu_t \]

- $\mathcal{H}_\infty$-norm: $\ell_2 \rightarrow \ell_2$ operator norm from $\{w\}$ to $\{z\}$
  \[ \|T(K)\|_\infty := \sup_{\theta \in [0,2\pi)} \lambda^{1/2}_{\text{max}} [T(K)(e^{-j\theta})]^\top T(K)(e^{j\theta}) \]
Bigger Picture: Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control

- **Solve**

  $$\min_{K} \mathcal{J}(K) \quad \text{s.t.} \quad \rho(A - BK) < 1 \quad \text{and} \quad \|T(K)\|_\infty < \gamma$$

- **$\mathcal{J}(K)$ upper-bounds $\mathcal{H}_2$-norm:**

  $$\mathcal{J}(K) = \text{Tr}(P_K DD^\top), \quad \text{or} \quad \mathcal{J}(K) = -\gamma^2 \log \det(I - \gamma^{-2}P_K DD^\top),$$

  where $P_K$ solves a **Riccati equation** given $K$

  $$P_K = \tilde{A}_K^T P_K \tilde{A}_K + \tilde{A}_K^T P_K D (\gamma^2 I - D^\top P_K D)^{-1} D^\top P_K \tilde{A}_K + C^\top C + K^\top R K$$

  with $\tilde{A}_K = A - BK$

- **If $\gamma \to \infty$, then $\mathcal{J}(K) \to \mathcal{H}_2$-norm, e.g., it reduces to LQR**
Why an Important & Interesting Model?

- Intuition: Small gain theorem – if $\|T(K)\|_\infty < \gamma$ and $\|\Delta\|_{\ell_2 \to \ell_2} < 1/\gamma$, then $G-\Delta$ is input-output stable
- Choosing

$$\mathcal{K} = \{K \mid \rho(A - BK) < 1; \|T(K)\|_\infty < \gamma\}$$

$\implies$ certain level of robust stability
  - $\gamma \to \infty$ reduces to stability region in LQR
- Second choice of $J(K)$ with $D = W^{1/2}$ and $\gamma = 1/\sqrt{\beta}$ coincides with the risk-sensitive LEQG objective
- It also unifies maximum-entropy/$\mathcal{H}_\infty$ control, LQG/$\mathcal{H}_\infty$ control [Glover and Doyle, ’88; Mustafa, ’89]; $\gamma$-level disturbance attenuation [Başar, ’91]; and zero-sum LQ dynamic games (come to it later) [Jacobson, ’73; Başar and Bernhard, ’95]. Also used for solving $\mathcal{H}_\infty$-optimal control
- Also used in Economics to model sequential decision-making under model uncertainty [Hansen and Sargent, ’08]
Algorithms and Landscape
Policy Gradient Algorithms

- Following the naming convention in [Fazel et al., '18] for LQR

**Policy Gradient:**

\[
K' = K - \eta \cdot \nabla J(K)
\]

\[
= K - 2\eta \cdot [(R + B^\top \tilde{P}_K B)K - B^\top \tilde{P}_K A] \cdot \Delta_K,
\]

**Natural PG:**

\[
K' = K - \eta \cdot \nabla J(K) \cdot \Delta^{-1}_K
\]

\[
= K - 2\eta \cdot [(R + B^\top \tilde{P}_K B)K - B^\top \tilde{P}_K A],
\]

**Gauss-Newton:**

\[
K' = K - \eta \cdot (R + B^\top \tilde{P}_K B)^{-1} \cdot \nabla J(K) \cdot \Delta^{-1}_K
\]

\[
= K - 2\eta \cdot [K - (R + B^\top \tilde{P}_K B)^{-1}B^\top \tilde{P}_K A]
\]

- Recall \( P_K \) is the solution to some Riccati equation dictated by \( K \), and \( \Delta_K \) is the solution to another (dual) Riccati equation
Landscape

**Lemma (Nonconvexity)**

There is a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem that is nonconvex for policy optimization.

**Lemma (Non-Coercivity)**

There is a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem whose cost function $J(K)$ is non-coercive. Particularly, as $K \to \partial \mathcal{K}$, $J(K)$ does not necessarily approach infinity.

- Coercivity is key in LQR analysis [Fazel et al., ’18][Bu et al., ’19][Malik et al., ’19][Mohammadi et al., ’19]
  - Ensures any descent direction to be feasible
  - Can confine everything to the sub-level set (reduces to standard smooth optimization)
Convergence
Descent in $\mathcal{J}(K)$ does not necessarily ensure feasibility/robust stability

How to enforce $K \in \mathcal{K}$ during iteration?
Implicit Regularization

- Regularization: \textit{iterate $K$ remains inside $\mathcal{K}$}, i.e., robustly stable
- Can be made explicit via \textit{projection onto $\mathcal{K}$}. But $\mathcal{K}$ is nonconvex, and defined in \textit{frequency domain}
- \textbf{Implicit} regularization:
  - The convergence of \textit{certain algorithms} behaves as if \textit{certain regularization} is used
  - Borrowed from machine learning literature: observed in learning overparametrized neural nets/nonlinear models [Kubo et al., ’19][Azizan et al., ’19], phase retrieval and matrix completion [Ma et al., ’17], etc., with (stochastic) gradient (mirror) descent
  - Property of both the \textit{nonconvex problem and algorithm}
- Gauss-Newton and natural PG enjoy implicit regularization!
Theory: Implicit Regularization

Theorem (Implicit Regularization)

Suppose the stepsizes $\eta$ satisfy:

- Gauss-Newton: $\eta \leq 1/2$,
- Natural policy gradient: $\eta \leq 1/(2\|R + B^T \tilde{P}_{K_0} B\|)$.

Then, $K \in \mathcal{K} \implies K' \in \mathcal{K}$.

- General descent directions of $\mathcal{J}(K)$ may not work, but certain directions do.
Implicit Regularization: Proof Idea

- Linear matrix inequalities (LMIs)-based approach
- A new use of Bounded Real Lemma [Başar & Bernhard, ’95][Zhou et al., ’96][Dullerud & Paganini, ’00]:
  \[ K \in \mathcal{K} \iff \text{Riccati equation} \iff \text{strict Riccati inequality (RI)} \]
- Observation: two consecutive iterates \( K \rightarrow K' \) are related – previous \( P_K \) is a candidate for the non-strict RI under \( K' \) to hold
- Perturb \( P_K \) in a certain way \( \Rightarrow \) strict RI
  - Perturb \( P = P_K + \alpha \bar{P} \) for small enough \( \alpha > 0 \), where \( \bar{P} > 0 \) solves the Lyapunov equation as before, with \( C^\top C + K^\top RK \) replaced by \(-I\)
Theory: Global Convergence

Theorem (Global Convergence)

Suppose $K_0 \in \mathcal{K}$, then both the Gauss-Newton and natural PG updates converge to the global optimum $K^* = (R + B^\top \tilde{P}_K B)^{-1} B^\top \tilde{P}_K A$ with $O(1/N)$ rate.
Theory: Local (Super-)Linear Rates

Much faster rates around the global optimum

Theorem (Local Faster Rates)

Suppose the conditions above hold, and additionally $DD^\top > 0$. Then, both the Gauss-Newton and natural PG updates converge to the optimal control gain $K^*$ with locally linear rate. In addition, if $\eta = 1/2$, the Gauss-Newton update converges to $K^*$ with locally Q-quadratic rate.

- Gradient domination (Polyak-Łojasiewicz) property holds locally
- Q-quadratic rate mirrors that of policy iteration for LQR [Lanchaster and Rodman ’95]
Simulations
Simulations

- Initialization near the boundary $\partial K$; infinitesimal stepsize $\eta$ for PG

Ave. Grad. Norm Square

$\mathcal{J}(K) - \mathcal{J}(K^*)$

$\mathcal{H}_\infty$-norm $\|T(K)\|_\infty$
Simulations: Global Convergence

- Escaping suboptimal stationary points
Simulations: Scalability

- Computationally more efficient than existing general robust control solvers – HIFOO [Arzelier et al., ’11] & Matlab h2hinfsyn function [Mahmoud and Pascal, ’96] and systune function [Apkarian et al., ’08]

<table>
<thead>
<tr>
<th>System Dim.</th>
<th>HIFOO</th>
<th>h2hinfsyn</th>
<th>systune</th>
<th>NPG</th>
<th>GN</th>
<th>Speedup</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 × 15</td>
<td>0.3742s</td>
<td>95.2663s</td>
<td>0.4276s</td>
<td>0.0481s</td>
<td>0.0420s</td>
<td>8/2117/8.8×</td>
</tr>
<tr>
<td>60 × 60</td>
<td>18.4380s</td>
<td>fail, &gt; 7200s</td>
<td>171.7855s</td>
<td>0.3906s</td>
<td>0.3902s</td>
<td>47/18461/440×</td>
</tr>
<tr>
<td>90 × 90</td>
<td>241.4416s</td>
<td>fail, &gt; 7200s</td>
<td>4126.9s</td>
<td>0.8167s</td>
<td>0.8103s</td>
<td>295/36922/5093×</td>
</tr>
</tbody>
</table>

Table: Average runtime comparison
Connection to Multi-Agent RL (MARL)
Multi-Agent RL

- Usually studied under framework of Markov games [Shapley ‘53]
- The most basic MARL model ever since [Littman, ’94]: two-player zero-sum Markov games
Multi-Agent RL

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- Benchmark in continuous control: linear quadratic zero-sum dynamic games (DG) [Başar and Bernhard, ’95] (mirrors LQR for single-agent RL)
Multi-Agent RL

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- PO methods widely used in modern empirical MARL, while its convergence guarantees remain largely open
Multi-Agent RL

- Usually studied under framework of **Markov games** [Shapley ’53]
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- Benchmark in continuous control: **linear quadratic zero-sum dynamic games** (DG) [Başar and Bernhard, ’95] (mirrors LQR for single-agent RL)
- PO methods widely used in modern **empirical MARL**, while its convergence guarantees remain largely open
  - (Projected) PO for LQ zero-sum DGs [ZYB, ’19][Bu et al., ’19]
  - Negative/Non-convergence results for (multi-player) LQ general-sum DGs [Mazumdar, Ratliff, Jordan, and Sastry, ’19]
  - PG methods (and variants) for tabular zero-sum Markov games [Daskalakis et al., ’20][Zhao et al., ’21][Cen et al., ’21,’22]...
Multi-Agent RL

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  - PG methods (and variants) for tabular zero-sum Markov games [Daskalakis et al., ’20][Zhao et al., ’21][Cen et al., ’21,’22]...
- Nonconvex-nonconcave [ZYB, ’19][Daskalakis et al., ’20], PO can easily diverge if not designed carefully
LQ Zero-Sum Dynamic Games

- $\mathcal{H}_2/\mathcal{H}_\infty$ control is strongly tied to LQ zero-sum dynamic games
- Let $u_t = -Kx_t$ and $w_t = -Lx_t$ then solve:

$$J(K, L) := \mathbb{E}_{x_0 \sim D} \left\{ \sum_{t=0}^{\infty} [x_t^\top Q x_t + (Kx_t)^\top R^u(Kx_t) - (Lx_t)^\top R^v(Lx_t)] \right\}$$

Solve: $\min_K \max_L J(K, L) \iff \min_K J(K, L(K))$

with $x_{t+1} = Ax_t + Bu_t + Cw_t$

- For fixed $K$ (outer-loop), take max over $L$ (inner-loop), the Riccati equation becomes

the same Riccati equation as in $\mathcal{H}_2/\mathcal{H}_\infty$ control
Implication for MARL

- Previous results $\implies$ double-loop update provably works
  - Double-loop/nested-grad.: fix $K$ and improve $L$, then improve $K$
  - Aligned with the empirical tricks to stabilize nonconvex-nonconcave minimax opt. with timescale separation [Lin, Jin, & Jordan, ’18], as in training GANs [Heusel et al., ’18]

- Gives global convergence of PO in competitive MARL (zero-sum Markov/dynamic games)
Benefit from MARL: Model-Free $\mathcal{H}_2/\mathcal{H}_\infty$ Control

- Recall the policy gradient form

$$\nabla \mathcal{J}(K) = 2[(R + B^\top \tilde{P}_K B)K - B^\top \tilde{P}_K A] \Delta K,$$

while $\Delta K$ cannot be estimated from sampled trajectories.

- Instead solve the equivalent game using data
  - Build up a virtual adversary $w_t = -Lx_t$
  - Double-loop/nested-grad.: fix $K$ and improve $L$, then improve $K$

- Derivative-free methods for LQR [Fazel et al., ’18][Malik et al., ’19] cannot work directly
  - Non-coercive & only certain direction works $\implies$ no uniform margin
  - Caveat: quantities (cost, action space, control gain matrices) in the LQ setting are continuous, and can easily go unbounded!
  - Leads to no-global-smoothness + nonconvexity-nonconcavity

- Can be addressed under the unified LQ game formulation, for finite-horizon settings [ZZHB, ’21]
Illustration for Derivative-Free PO Convergence

- Proof idea illustrated with figures
  - $\mathcal{K}_0$ is the “level-set” corresponding to initialization $K_0$;
  - $\hat{\mathcal{K}}$ is a larger set that the iterates will not leave (with high probability); due to stochastic errors when using samples
  - both are compact $\implies$ uniform smoothness constant over $\hat{\mathcal{K}}$
Connection to Robust Adversarial RL (RARL)
Robust Adversarial RL [Pinto et al., ’17]

- RL hardly generalizes due to Sim2real and/or training-testing gap

- One remedy: RARL [Pinto et al., ’17]
  - Idea: introduce an adversary, playing against the agent
  - Dates back to [Morimoto and Doya, ’05], under the name robust RL, and “inspired by $\mathcal{H}_\infty$-theory”
  - Made popular by the empirical work [Pinto et al., ’17]
  - Question: Any robustness interpretation and convergence guarantee?
LQ RARL

- RARL setting $\iff$ zero-sum dynamic game
- LQ RARL: View $w_t$ as model-uncertainty, or the model-misspecification when linearizing a nonlinear model
- Recall the RARL scheme in [Pinto et al., ’17]

---

**Algorithm 1** Policy-Based LQ RARL Scheme (Pinto et al., 2017)

```
Input: LQ RARL environment, initial policies $(K_0, L_0)$
for $n = 1, \ldots, N$ do
    Update $L_n \leftarrow L_{n-1}$
    for $j = 1, \ldots, N_L$ do
        Update $L_n \leftarrow \text{PolicyOptimizer}(K_{n-1}, L_n)$
    end for
    Update $K_n \leftarrow K_{n-1}$
    for $i = 1, \ldots, N_K$ do
        Update $K_n \leftarrow \text{PolicyOptimizer}(K_n, L_n)$
    end for
end for
Return: policy pair $(K_N, L_N)$
```
RARL in [Pinto et al., ’17] Easily Fails [ZHB, ’20]

- Stability issue due to bad initialization
- Stability issue due to bad choices of $(N_K, N_L)$ $(K_0, L_0)$

- What is a good combination of initialization & update rule?
By implicit regularization, we find a provably convergent pair of (initialization, update rule): \((K_0 \in \mathcal{K}, L_0 = 0), \ (N_K = 1, \ N_L = \infty)\)

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Double-loop Update</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong></td>
<td>Initialize (K_0 \in \mathcal{K}, \ L_0) stabilizing, e.g., (L_0 = 0)</td>
</tr>
<tr>
<td>for (n = 0, \cdots) do</td>
<td></td>
</tr>
<tr>
<td>for (i = 0, \cdots) do</td>
<td></td>
</tr>
<tr>
<td>(L_{i+1} \leftarrow \text{PolicyOptimizer}(K_n, L_i))</td>
<td></td>
</tr>
<tr>
<td>end for</td>
<td></td>
</tr>
<tr>
<td>Update (K_{n+1} \leftarrow \text{PolicyOptimizer}(K_n, L_\infty))</td>
<td></td>
</tr>
<tr>
<td>end for</td>
<td></td>
</tr>
</tbody>
</table>
Implication from Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control

- By implicit regularization, we find a provably convergent pair of (initialization, update rule): $((K_0 \in \mathcal{K}, L_0 = 0), (N_K = 1, N_L = \infty))$
- How to find such a $K_0 \in \mathcal{K}$ (in a model-free way) – robustify $K_0$?
- For any stabilizing $K$, perform $K' = K - \alpha g$ with $g \in \mathbb{R}^{m \times n}$ the finite-difference estimate of the subgradient of $\|T(K)\|_\infty$, where

$$g_{ij} = \frac{\|T(K + \epsilon d_{ij})\|_\infty - \|T(K - \epsilon d_{ij})\|_\infty}{2\epsilon}$$

![Decrease $\mathcal{H}_\infty$-norm Before Robustification](image1)

![Before Robustification](image2)

![After Robustification](image3)
Additional Simulations
Convergent Cases

▶ Exact update:

▶ Derivative-free update:
Some Divergent Cases

- Update-rules other than **double-loop** may easily diverge, even with infinitesimal stepsizes
  - ANGDA: alternative-update of natural PG descent & ascent
  - $\tau$-NGDA: simultaneous-update with stepsizes ratio $\frac{\eta}{\alpha} = \tau > 1$
  - Descent-Multi-Step-Ascent: multiple ascent steps per descent step
Without global smoothness, controlling the iterates’ boundedness is critical and challenging.
Concluding Remarks
Concluding Remarks

- Studied policy optimization landscape for risk-sensitive/robust control, with fundamental challenges diff. from that of LQR – deepened our understanding of existing results on LQR
- Developed two PO methods, identified their implicit regularization property, and established global convergence + sample complexity
- Along the way
  - Global convergence and sample complexity of PO for competitive MARL, in the LQ zero-sum setting
  - Some theoretical understanding and critical thinking on RARL, from robust control perspective
  - Explicit regularization and convex-reformulation can also be useful — a unified differentiable convex liftings (DCL) framework [USPZT, ’22]
- A natural intersection of control, RL, and game theory
Thank You!
Direct Policy Search for Robust Control: A Nonsmooth Optimization Perspective

Bin Hu

ECE & CSL, University of Illinois Urbana-Champaign

L4DC Tutorial 2023
Joint work with Xingang Guo
Outline

• Motivation and Problem Formulation

• Main Results

• Conclusions and Future Directions
Motivation: Reinforcement Learning for Control

- Many robust control problems are solved via lifting into convex spaces. Recently, reinforcement learning (RL) has shown great promise for control!

- Main workhorse: direct policy search/policy optimization (PO)
  \[
  \min_{K} \ J(K), \quad \text{s.t.} \ K \in \mathcal{K}
  \]
  - Parametrized policy $K$ (e.g. linear mapping, neural networks)
  - Cost function $J$ (tracking errors, closed-loop $\mathcal{H}_2/\mathcal{H}_\infty$ norms, etc)
  - Constraint set $\mathcal{K}$ (stability, robustness, safety, etc)
  - PO algorithm: $K' = K - \alpha \nabla J(K)$ (nonconvex problem!)
  - Theory: Landscape, feasibility, convergence, complexity

- Question: How to tailor policy-based RL for robust control?
- This talk: Guarantees of PO on $\mathcal{H}_\infty$ control benchmarks
PO Theory for Robust Control

- **PO theory for mixed design (maintaining robustness via improving average)**
  - Landscape: Feasible set is connected, and stationary is global
  - Feasibility: The cost is nonconvex and non-coercive! Fortunately, double-loop natural policy gradient (NPG) can **implicitly regularize**
  - Global sublinear convergence for NPG
  - Ref: Zhang, Hu, Başar. Policy optimization for $\mathcal{H}_2$ linear Control with $\mathcal{H}_\infty$ robustness guarantee: Implicit regularization and global convergence, SICON 2021.

- **PO theory for $\mathcal{H}_\infty$ state-feedback synthesis (improving robustness)**
  - Feature: Nonconvex nonsmooth
  - Landscape: Any Clarke stationary points are global
  - Feasibility: The cost is coercive and serves as a barrier function on $\mathcal{K}$
  - Global convergence: Goldstein’s subgradient method achieves global convergence provably
Review: Linear Quadratic Regulator

- LQR as PO: Consider $x_{t+1} = Ax_t + Bu_t + w_t$ with $w_t$ being stochastic IID

$$\min_K J(K) := \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} (x_t^\top Q x_t + u_t^\top R u_t) \right], \quad \text{s.t. } K \text{ is stabilizing}$$

- $u_t = -K x_t$ for gain matrix $K$
- $\mathcal{K} = \{K | \rho(A - BK) < 1\}$; $\mathcal{K}$ a nonconvex constraint set

- PO theory for LQR
  - Landscape: Stationary is global
  - Feasibility: The LQR cost is coercive and serves as a barrier on $\mathcal{K}$
  - Global convergence & sample complexity: Linear rate via the gradient dominance/smoothness property

- Main Ref:
  
Problem Formulation: State-feedback $\mathcal{H}_\infty$ Control

Consider the following linear time-invariant (LTI) system

$$x_{t+1} = Ax_t + Bu_t + w_t, \quad x_0 = 0.$$ 

- We assume that $(A, B)$ is stabilizable
- $u := \{u_0, u_1, \cdots\}$, $w := \{w_0, w_1, \cdots\}$, and $\|w\| = (\sum_{t=0}^{\infty} \|w_t\|^2)^{1/2}$.
- Our goal is to find a sequence $u$ to minimize the quadratic cost function

$$\min_{u} \max_{w: \|w\| \leq 1} \sum_{t=0}^{\infty} (x_t^T Q x_t + u_t^T R u_t)$$

in the presence of the worst case disturbance $\|w\| \leq 1$.
- This is different than the LQR problem, where $w$ is stochastic.
- $\|w\| \leq 1$ is not restrictive, we can choose arbitrary $\ell_2$ bound.
- We assume that $Q$ and $R$ are positive definite.
- It is well known that the optimal solution is using a linear state-feedback policy $u_t = -K x_t$ (Başar and Bernhard 2008).
Problem Formulation: State-feedback $\mathcal{H}_\infty$ Control

Consider $u_t = -Kx_t$, the closed-loop system becomes $x_{t+1} = (A - BK)x_t + w_t$. We have the following PO problem:

$$\min_K \max_{w: \|w\| \leq 1} \sum_{t=0}^{\infty} x_t^T (Q + K^T R K) x_t.$$  

The above optimization problem equivalent to the following PO problem

$$\min_K J(K) := \sup_{\omega \in [0, 2\pi]} \sigma_{\max}((Q + K^T R K)^{1/2}(e^{j\omega} I - A + BK)^{-1})$$

s.t. $K \in \mathcal{K} := \{K : \rho(A - BK) < 1\}$.

- This is a constrained nonconvex nonsmooth optimization problem.
- $\mathcal{K}$ can be nonconvex.
- The nonsmoothness comes from two sources:
  1. The computation of the maximum singular value.
  2. The operator $\sup$ over $\omega \in [0, 2\pi]$. 
Convex LMIs vs. Direct Policy Search

• In 1980s, convex optimization methods become popular for control study due to global guarantees and efficient interior point methods

• Reparameterize problems as convex optimization problems (one does not optimize the controller parameters directly)

\[
\{ K \in \mathcal{K} : J(K) \leq \gamma \} \\
\iff \{ K = LY^{-1} : \text{LMI}(Y, L, \gamma) \preceq 0 \text{ is feasible, } Y \succ 0 \}.
\]

• Minimizing \( \gamma \) over \( \text{LMI}(Y, L, \gamma) \preceq 0 \) and \( Y \succ 0 \) is a SDP problem

• See for example, Boyd et al., “Linear Matrix Inequalities in System and Control Theory”, 1994, SIAM

• PO is not convex!

• In this past, PO has been a popular approach for problems that cannot be convexified, e.g. structured \( \mathcal{H}_\infty \) synthesis! (HIFOO and Hinfstrcuct)

• This talk: View \( \mathcal{H}_\infty \) synthesis as a benchmark for understanding PO
Some Background on Nonsmooth Optimization

Clarke subdifferential:

\[
\partial_C J(K) := \text{conv}\{ \lim_{i \to \infty} \nabla J(K_i) : K_i \to K, K_i \in \text{dom}(\nabla J) \subset K\}.
\]

- \(\partial_C J(K)\) is well defined for any \(K \in \mathcal{K}\).
- \(J(K)\) is locally Lipschitz and hence almost everywhere differentiable.

**Proposition**

If \(K\) is a local minimum of \(J\), then \(0 \in \partial_C J(K)\) and \(K\) is a Clarke stationary point.
Some Background on Nonsmooth Optimization

Generalized Clarke directional derivative:

\[ J^\circ(K, d) := \lim_{K' \to K} \sup_{t \downarrow 0} \frac{J(K' + td) - J(K')}{t}. \]

Directional derivative:

\[ J'(K, d) := \lim_{t \downarrow 0} \frac{J(K + td) - J(K)}{t}. \]

- \( J^\circ(K, d) \) and \( J'(K, d) \) are different in general.
- \( J'(K, d) = J^\circ(K, d) \) if \( J(K) \) is subdifferentially regular.

Subdifferentially Regular Property

Let \( K^\dagger \) be a Clarke stationary point for \( J \). If \( J \) is subdifferentially regular, then \( J'(K^\dagger, d) \geq 0 \) for all \( d^a \).

\[^a\text{This result is known, see Theorem 10.1 in Rockafellar and Wets 2009.}\]
Some Background on Nonsmooth Optimization

Goldstein subdifferential:

$$\partial_\delta J(K) := \text{conv} \left\{ \bigcup_{K' \in \mathbb{B}_\delta(K)} \partial C J(K') \right\},$$

- $\mathbb{B}_\delta(K)$ is the $\delta$-ball around $K$
- requires $\mathbb{B}_\delta(K) \subset \mathcal{K}$.

Generating a good descent direction (Goldstein1977):

**Descent inequality**

Let $F$ be the minimal norm element in $\partial_\delta J(K)$. Suppose $K - \alpha F/\|F\|_2 \in \mathcal{K}$ for any $0 \leq \alpha \leq \delta$. Then we have:

$$J(K - \delta F/\|F\|_2) \leq J(K) - \delta \|F\|_2.$$
Outline

• Motivation and Problem Formulation

• Main Results

• Conclusions and Future Directions
Summary of Known Facts

\[
\min_K J(K) := \sup_{\omega \in [0, 2\pi]} \sigma_{\text{max}} \left( (Q + K^T R K)^{1/2} (e^{i\omega} I - A + BK)^{-1} \right)
\]

s.t. \( K \in \mathcal{K} := \{ K : \rho(A - BK) < 1 \} \).

- \( \mathcal{K} \) is open, can be unbounded, and nonconvex.
- \( J(K) \) is continuous, nonsmooth, and can be nonconvex in \( K \).
- \( J(K) \) is locally Lipschitz, subdifferentially regular over the feasible set \( \mathcal{K} \).
High Level Ideas

Goldstein’s subgradient method:

\[ K^{n+1} = K^n - \delta^n F^n / \| F^n \|_2, \]

- \( F^n \) is the minimum norm element of \( \partial \delta_n J(K^n) \).
- \( K^0 \in \mathcal{K} \) is known.

High level ideas:

- Goldstein’s subgradient method finds Clarke stationary point
- Coerciveness ensures \( K^n \) stay within the nonconvex feasible set.
- Clarke stationary points are global, and hence global optimum is found.
Main Results

**Theorem (Guo and Hu, NeurIPS2022)**

Suppose $(Q, R)$ are positive definite, and $(A, B)$ is stabilizable. We have

1. $J(K)$ is coercive over the set $\mathcal{K}$. (*Proved via the properties of $(Q, R)$*)
2. For any $K \in \mathcal{K}$ satisfying $J(K) > J^*$, there exists $V \neq 0$ s.t. $J'(K, V) < 0$.
3. Any Clarke stationary points of the $H_\infty$ cost are global minimum.
4. For any $\gamma > J^*$, the sublevel set $\mathcal{K}_\gamma = \{ K \in \mathcal{K} : J(K) \leq \gamma \}$ is compact. There is a strict separation between $\mathcal{K}_\gamma$ and $\mathcal{K}^c$.
5. Suppose $K^0 \in \mathcal{K}$. Set $\Delta_0 := \text{dist}(\mathcal{K}_{J(K^0)}, \mathcal{K}^c) > 0$, and $\delta^n = \frac{0.99 \Delta_0}{n+1}$. Then Goldstein’s subgradient method $K^{n+1} = K^n - \delta^n F^n / \| F^n \|_F$ with $F^n$ being the minimum norm element of $\partial \delta^n J(K^n)$ is guaranteed to stay in $\mathcal{K}$ for all $n$. In addition, we have $J(K^n) \to J^*$ as $n \to \infty$.
6. There is also a complexity result for finding $(\varepsilon, \delta)$-stationary points.

The most technical part of the proof is for Step 2. It requires the use of non-strict version of the KYP lemma.
Step 2 of Main Result

Lemma

For any $K \in \mathcal{K}$ that $J(K) > J^*$, there exists a direction $d \neq 0$ such that the directional derivative $J'(K, d) \leq J^* - J(K) < 0$.

Proof Sketch:

By convexity, we have

$$LMI(Y + t\Delta Y, L + t\Delta L, \gamma + t(\gamma^* - \gamma)) \leq 0$$

with $\Delta Y = Y^* - Y$, $\Delta L = L^* - L$, and $t \in [0, 1]$. In addition, we must have

$$J(((L + t\Delta L)(Y + t\Delta Y)^{-1}) \leq \gamma + t(\gamma^* - \gamma).$$
Lemma
For any $K \in \mathcal{K}$ that $J(K) > J^*$, there exists a direction $d \neq 0$ such that the directional derivative $J'(K, d) \leq J^* - J(K) < 0$.

Proof Sketch Con:
Based on the fact $J(K^*) < J(K)$, we can construct a direction $d$ such that $J'(K, d) < 0$. Specifically, consider $d = \Delta LY^{-1} - LY^{-1} \Delta YY^{-1}$. Then we have

$$J'(K, d) = \lim_{t \downarrow 0} \frac{J(K + t(\Delta LY^{-1} - LY^{-1} \Delta YY^{-1})) - J(K)}{t}$$

$$\leq \lim_{t \downarrow 0} \left( \frac{J((L + t\Delta L)(Y + t\Delta Y)^{-1}) - J(K)}{t} + O(t) \right)$$

$$\leq \lim_{t \downarrow 0} \left( \frac{J(K) + t(J(K^*) - J(K)) - J(K)}{t} + O(t) \right)$$

$$= J(K^*) - J(K) < 0,$$

we use the fact that $(Y + t\Delta Y)^{-1} = Y^{-1} - tY^{-1} \Delta YY^{-1} + O(t^2)$. ■
Finite-time complexity for \((\delta, \varepsilon)\)-stationary points

Goldstein’s subdifferential: \(\partial_{\delta} J(K) := \text{conv} \left\{ \cup_{K' \in B_{\delta}(K)} \partial_C J(K') \right\} \).

**Definition**

A point \(K\) is said to be \((\delta, \varepsilon)\)-stationary if \(\text{dist}(0, \partial_{\delta} J(K)) \leq \varepsilon\).

**Theorem 3**

If we choose \(\delta^n = \delta < \Delta_0\), then we have:

- \(K^n \in \mathcal{K}\) for all \(n\)
- \(\min_{0 \leq n \leq N} \|F^n\|_2 \leq \frac{J(K^0) - J^*}{(N+1)\delta}\), i.e., the complexity of finding a \((\delta, \varepsilon)\)-stationary point is \(O\left(\frac{\Delta}{\delta \varepsilon}\right)\)

- \((\delta, \varepsilon)\)-stationarity does not imply being \(\delta\)-close to an \(\varepsilon\)-stationary point of \(J\).
- Finite time bounds for \((J(K^n) - J^*)\) is possible via exploiting other advanced properties \(J(K)\).
Implementable Algorithms

Finding minimum norm element of Goldstein’s subdifferential may not be easy. Fortunately, there are many implementable variants:

- **Gradient Sampling (GS)** (The HIFOO toolbox): The main idea is to randomly generate differentiable samples over $\mathbb{B}_{\delta n}(K^n)$ with probability 1. The convex hull of the gradients of these samples can be used as an approximation of $\partial_{\delta n} J(K^n)$.

- **Nonderivative Sampling (NS)** (Kiwiel2010): The NS method can be viewed as the derivative-free version of the GS algorithm by only using the zeroth-order oracle.

- **Interpolated normalized gradient descent (INGD)** (Zhang, J., et al. 2020; Davis, D., et al. 2022): INGD uses an iterative sampling strategy to generate a descent direction. The INGD algorithm is guaranteed to find the $(\delta, \varepsilon)$-stationary point with the high-probability finite-time complexity bound.
Numerical Example

To support our theory, we provide some numerical simulations. Consider the following example:

\[
A = \begin{bmatrix} 1 & 0 & -5 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad R = 1.
\]

For this example, we have \( J^* = 7.3475 \). We initialize from

\[
K^0 = \begin{bmatrix} 0.4931 & -0.1368 & -2.2654 \end{bmatrix},
\]

which satisfies \( \rho(A - BK^0) = 0.5756 < 1 \).
Numerical Example

Figure: Simulation results. Left: The trajectory of relative error of GS, NS, INGD, and Model-free NS methods. Middle: The trajectory of the relative optimality gap of 8 randomly generated cases for the NS method. Right: The trajectory of the Model-free NS method with more noisy oracle.
Outline

• Motivation and Problem Formulation

• Main Results

• Conclusions and Future Directions
Take Aways

We studied the global convergence of direct policy search on state-feedback $\mathcal{H}_\infty$ robust control synthesis.

- State-feedback $\mathcal{H}_\infty$ synthesis is a constrained nonconvex nonsmooth policy optimization problem.

- Any Clarke stationary points for this problem are actually global minimum.

- Goldstein’s subgradient methods are guaranteed to stay within the nonconvex feasible set and converge to the global optimal.

- $(\delta, \varepsilon)$-stationary points can be found with finite-time guarantees.

- $(\delta, \varepsilon)$-stationarity does not imply being $\delta$-close to an $\varepsilon$-stationary point of $J$. 
Future Work

• Finite-time bounds for the optimality gap (i.e. $J(K^n) - J^*$)

• The sample complexity of direct policy search on model-free $\mathcal{H}_\infty$ control

• Other $\mathcal{H}_\infty$ synthesis problems (static/dynamic output feedback, etc)
Thanks!

If you are interested, feel free to send an email to binhu7@illinois.edu

Funding & Support: NSF
Analysis of the Optimization Landscape of Linear Quadratic Gaussian (LQG) Control

Work by Yang Zheng, Yujie Tang, and Na Li

Presented by Bin Hu

5th Annual Learning for Dynamics & Control Conference
University of Pennsylvania. June 14-16, 2023
Today’s talk

**Optimal Control**

**Feedback Paradigm**

- Control input $d(t)$, $u(t)$
- Measurement $w(t)$, $y(t)$
- System $x(t)$

**Linear Quadratic Optimal control**

$$\min_{u_1, u_2, \ldots} \lim_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^{T} (x_t^T Q x_t + u_t^T R u_t) \right]$$

subject to

$$x_{t+1} = Ax_t + Bu_t + w_t$$

$$y_t = Cx_t + v_t$$

- Many practical applications
- **Linear Quadratic Regulator (LQR)** when the state $x_t$ is directly observable
- **Linear Quadratic Gaussian (LQG) control** when only partial output $y_t$ is observed
- Extensive classical results (Dynamic programming, Separation principle, Riccati equations, etc.)

They are all model-based. Are there any guarantees for non-convex policy optimization?
Challenges for partially observed LQG

Policy optimization for LQG control

- LQG is more complicated than LQR
- Requires dynamical controllers
- Its non-convex landscape properties are much richer and more complicated than LQR

Our focus: non-convex LQG landscape

Q1: Properties of the domain (set of stabilizing controllers)
- convexity, connectivity, open/closed?

Q2: Properties of the accumulated LQG cost
- convexity, differentiability, coercivity?
- set of stationary points/local minima/global minima?
Outline

❑ LQG problem Setup

❑ Connectivity of the Set of Stabilizing Controllers

❑ Structure of Stationary Points of the LQG cost
LQG Problem Setup

**Objective:** The LQG cost

\[
\lim_{T \to +\infty} \frac{1}{T} \mathbb{E} \int_0^T (x^\top Q x + u^\top R u) \, dt
\]

- \( \xi(t) \) internal state of the controller
- \( \dim \xi(t) \) order of the controller
- \( \dim \xi(t) = \dim x(t) \) full-order
- \( \dim \xi(t) < \dim x(t) \) reduced-order

**Minimal controller**

The input-output behavior cannot be replicated by a lower order controller.

*\((A_K, B_K, C_K)\) controllable and observable

---

**Plant**

\[
\begin{align*}
\dot{x}(t) &= A x(t) + B u(t) + w(t) \\
y(t) &= C x(t) + v(t)
\end{align*}
\]

**Dynamical controller**

\[
\begin{align*}
\dot{\xi}(t) &= A_K \xi(t) + B_K y(t) \\
u(t) &= C_K \xi(t)
\end{align*}
\]

**Standard Assumption**

- \((A, B), (A, W^{1/2})\) Controllable
- \((C, A), (Q^{1/2}, A)\) Observable

**Gaussian white**

- \(v(t)\)
- \(w(t)\)

---

LQG Problem Setup
Separation principle

**Gaussian white**

Plant:
\[
\dot{x}(t) = Ax(t) + Bu(t) + w(t) \\
y(t) = Cx(t) + v(t)
\]

\[
\dot{\xi}(t) = A_K \xi(t) + B_K y(t) \\
u(t) = C_K \xi(t)
\]

**Objective:** The LQG cost
\[
\lim_{T \to +\infty} \frac{1}{T} \mathbb{E} \int_0^T (x^\top Q x + u^\top R u) \, dt
\]

**Solution:** Kalman filter for state estimation
\[
\dot{\xi} = (A - BK)\xi + L(y - C\xi). \\
u = -K\xi.
\]

Two Riccati equations

- **Kalman gain** \( L = PC^\top V^{-1} \)

- **Feedback gain** \( K = R^{-1}B^\top S \)

Explicit dependence on the dynamics
Policy Optimization formulation

- **Closed-loop dynamics**
  \[
  \frac{d}{dt} \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} A & BC_K \\ B_KC & A_K \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix},
  \]
  \[
  \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & C_K \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix}.
  \]

- **Feasible region of the controller parameters**
  \[C_{\text{full}} = \left\{ K \mid K = (A_K, B_K, C_K) \text{ is full order}, \begin{bmatrix} A & BC_K \\ B_KC & A_K \end{bmatrix} \text{ is Hurwitz stable} \right\}\]

- **Cost function**
  \[
  J(K) = \lim_{T \to +\infty} \frac{1}{T} \mathbb{E} \int_0^T (x^T Q x + u^T R u) \, dt
  \]
  \[
  J(K) = \text{tr} \left( \begin{bmatrix} Q & 0 \\ 0 & C_K^T R C_K \end{bmatrix} X_K \right) = \text{tr} \left( \begin{bmatrix} W & 0 \\ 0 & B_K V B_K^T \end{bmatrix} Y_K \right)
  \]

  \[X_K, Y_K \quad \text{Solution to Lyapunov equations}\]

**Policy optimization for LQG**

\[
\min_K J(K) \quad \text{s.t.} \quad K = (A_K, B_K, C_K) \in C_{\text{full}}
\]

**Direct policy search**

\[K_{i+1} = K_i - \alpha_i \nabla J(K_i)\]

- **Does it converge at all?**
- **Converge to which point?**
- **Convergence speed?**

Main questions

- Q1: Connectivity of the feasible region $C_{\text{full}}$
  - Is it connected?
  - If not, how many connected components can it have?
- Q2: Structure of stationary points of $J(K)$
  - Are there spurious (strictly suboptimal, saddle) stationary points?
  - How to check if a stationary point is globally optimal?

Policy optimization for LQG

$$\min_{K} \quad J(K)$$
$$\text{s.t.} \quad K = (A_K, B_K, C_K) \in C_{\text{full}}$$

Non-convex Landscape Analysis
Outline

- LQG problem Setup
- Connectivity of the Set of Stabilizing Controllers
- Structure of Stationary Points of the LQG cost
Connectivity of the feasible region

❑ Simple observation: non-convex and unbounded

Lemma 1: the set $C_{\text{full}}$ is non-empty, unbounded, and can be non-convex.

Example

\[
\begin{align*}
\dot{x}(t) &= x(t) + u(t) + w(t) \\
y(t) &= x(t) + v(t)
\end{align*}
\]

\[
C_{\text{full}} = \left\{ K = \begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix} \in \mathbb{R}^{2 \times 2} \middle| \begin{bmatrix} 1 \\ B_K \\ A_K \end{bmatrix} \text{ is stable} \right\}.
\]

\[
K^{(1)} = \begin{bmatrix} 0 & 2 \\ -2 & -2 \end{bmatrix}, \quad K^{(2)} = \begin{bmatrix} 0 & -2 \\ 2 & -2 \end{bmatrix}
\]

Stabilize the plant, and thus belong to $C_{\text{full}}$

\[
\hat{K} = \frac{1}{2} \left( K^{(1)} + K^{(2)} \right) = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}
\]

Fails to stabilize the plant, and thus outside $C_{\text{full}}$
Main Result 1: dis-connectivity

**Theorem 1:** The set $C_{\text{full}}$ can be disconnected but has at most 2 connected components.

- Different from the connectivity of static stabilizing state-feedback controllers, which is always connected!
- Is this a negative result for gradient-based algorithms?  → **No**
Connectivity of the feasible region

Main Result 2: dis-connectivity

**Theorem 2:** If $C_{\text{full}}$ has 2 connected components, then there is a smooth bijection $T$ between the 2 connected components that has the same cost function value.

In fact, the bijection $T$ is defined by a similarity transformation (change of controller state coordinates)

$$T(K) := \begin{bmatrix} D_K & C_K T^{-1} \\ TB_K & TA_K T^{-1} \end{bmatrix}.$$

Positive news: For gradient-based local search methods, it makes no difference to search over either connected component.
Connectivity of the feasible region

Main Result 3: conditions for connectivity

**Theorem 3:**
1) \( C_{\text{full}} \) is connected if there exists a reduced-order stabilizing controller.
2) The sufficient condition above becomes necessary if the plant is single-input or single-output.

**Corollary 1:** Given any open-loop stable plant, the set of stabilizing controllers \( C_{\text{full}} \) is connected.

**Example: Open-loop stable system**

\[
\dot{x}(t) = -x(t) + u(t) + w(t) \\
y(t) = x(t) + v(t)
\]

**Routh--Hurwitz stability criterion**

\[
C_{\text{full}} = \left\{ K = \begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix} \in \mathbb{R}^{2 \times 2} \mid A_K < 1, B_K C_K < -A_K \right\}.
\]
Connectivity of the feasible region

Main Result 3: conditions for connectivity

Example: Open-loop unstable system (SISO)

\[
\begin{align*}
\dot{x}(t) &= x(t) + u(t) + w(t) \\
y(t) &= x(t) + v(t)
\end{align*}
\]

• Routh–Hurwitz stability criterion

\[
C_{\text{full}} = \left\{ K = \begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix} \in \mathbb{R}^{2 \times 2} \mid \begin{bmatrix} A & BC_K \\ B_KC & A_K \end{bmatrix} \text{ is stable} \right\}
\]

\[
= \left\{ K = \begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix} \in \mathbb{R}^{2 \times 2} \mid A_K < -1, \ B_KC_K < A_K \right\}.
\]

• Two path-connected components

\[
C_1^+ := \left\{ K = \begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix} \in \mathbb{R}^{2 \times 2} \mid A_K < -1, \ B_KC_K < A_K, \ B_K > 0 \right\},
\]

\[
C_1^- := \left\{ K = \begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix} \in \mathbb{R}^{2 \times 2} \mid A_K < -1, \ B_KC_K < A_K, \ B_K < 0 \right\}.
\]
Policy Optimization formulation

Q1: Connectivity of the feasible region $C_{\text{full}}$
- Is it connected? No
- If not, how many connected components can it have? Two

Q2: Structure of stationary points of $J(K)$
- Are there spurious (strictly suboptimal, saddle) stationary points?
- How to check if a stationary point is globally optimal?
Outline

- LQG problem Setup
- Connectivity of the Set of Stabilizing Controllers
- Structure of Stationary Points of the LQG cost
Structure of Stationary Points

- **Simple observations**
  1) $J(K)$ is a real analytic function over its domain (smooth, infinitely differentiable)
  2) $J(K)$ has **non-unique** and **non-isolated** global optima

\[
\dot{\xi}(t) = A_K \xi(t) + B_K y(t) \\
u(t) = C_K \xi(t)
\]

**Similarity transformation**

\[(A_K, B_K, C_K) \rightarrow (TA_K T^{-1}, TB_K, C_K T^{-1})\]

- $J(K)$ is invariant under similarity transformations.
- It has many stationary points, unlike the LQR with a unique stationary point

---

**Policy optimization for LQG**

\[
\begin{align*}
\min_{K} & \quad J(K) \\
\text{s.t.} & \quad K = (A_K, B_K, C_K) \in C_{\text{full}}
\end{align*}
\]
Structure of Stationary Points

Gradient computation

Lemma 2: For every $K = (A_K, B_K, C_K) \in C_{\text{full}}$, we have

\[
\frac{\partial J(K)}{\partial A_K} = 2 \left( Y_{12}^T X_{12} + Y_{22} X_{22} \right),
\]

\[
\frac{\partial J(K)}{\partial B_K} = 2 \left( Y_{22} B_K V + Y_{22} X_{12}^T C^T + Y_{12}^T X_{11} C^T \right),
\]

\[
\frac{\partial J(K)}{\partial C_K} = 2 \left( R C_K X_{22} + B^T Y_{11} X_{12} + B^T Y_{12} X_{22} \right),
\]

where $X_K = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$, $Y_K = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}$

are the unique positive semidefinite solutions to two Lyapunov equations.

How does the set of Stationary Points look like?

\[
\begin{align*}
K \in C_{\text{full}} & : \\
\frac{\partial J(K)}{\partial A_K} &= 0, \\
\frac{\partial J(K)}{\partial B_K} &= 0, \\
\frac{\partial J(K)}{\partial C_K} &= 0,
\end{align*}
\]

- Non-unique, non-isolated
- Local minimum, local maximum, saddle points, or globally minimum?
Structure of Stationary Points

Main Result: existences of strict saddle points

**Theorem 4:** Consider any open-loop stable plant. The zero controller with any stable $A_K$

$$K = (A_K, 0, 0) \in C_{\text{full}}$$

is a stationary point. Furthermore, the corresponding hessian is either indefinite (**strict saddle point**) or equal to zero (**high-order saddle or else**).

Example:

$$\dot{x}(t) = -x(t) + u(t) + w(t) \quad Q = 1, R = 1, V = 1, W = 1$$

$$y(t) = x(t) + v(t)$$

Stationary point: $K^* = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \in \mathbb{R}^{2 \times 2}, \quad \text{with } a < 0$

Cost function:

$$J \left( \begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix} \right) = \frac{A_K^2 - A_K(1 + B_K^2 C_K^2) - B_K C_K(1 - 3 B_K C_K + B_K^2 C_K^2)}{2(-1 + A_K)(A_K + B_K C_K)}.$$

Hessian:

$$\begin{bmatrix}
\frac{\partial J^2(K)}{\partial A_K^2} & \frac{\partial J^2(K)}{\partial A_K B_K} & \frac{\partial J^2(K)}{\partial A_K C_K} \\
\frac{\partial J^2(K)}{\partial B_K A_K} & \frac{\partial J^2(K)}{\partial B_K^2} & \frac{\partial J^2(K)}{\partial B_K C_K} \\
\frac{\partial J^2(K)}{\partial C_K A_K} & \frac{\partial J^2(K)}{\partial C_K B_K} & \frac{\partial J^2(K)}{\partial C_K^2}
\end{bmatrix} \bigg|_{K^*} = \frac{1}{2(1 - a)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

Indefinite with eigenvalues:

0 and $\pm \frac{1}{2(1-a)}$
Structure of Stationary Points

Main Result: existences of strict saddle points

**Theorem 4:** Consider any open-loop stable plant. The zero controller with any stable $A_K$

$$K = (A_K, 0, 0) \in C_{\text{full}}$$

is a stationary point. Furthermore, the corresponding hessian is either indefinite (strict saddle point) or equal to zero (high-order saddle or else).

Another example with zero Hessian

How does the set of Stationary Points look like?

$$\begin{align*}
K \in C_{\text{full}} & \quad \frac{\partial J(K)}{\partial A_K} = 0, \\
& \quad \frac{\partial J(K)}{\partial B_K} = 0, \\
& \quad \frac{\partial J(K)}{\partial C_K} = 0,
\end{align*}$$

- Non-unique, non-isolated
- Strictly suboptimal points; Strict saddle points
- All bad stationary points correspond to non-minimal controllers
Structure of Stationary Points

Main Result

Theorem 5: All stationary points corresponding to controllable and observable controllers are globally optimum.

 Particularly, given a stationary point that is a minimal controller

1) It is globally optimal, and the set of all global optima forms a manifold with 2 connected components.

Local Zero Gradient + Structural Information → Global Optimality Certificate

Example: open-loop unstable system
\[
\dot{x}(t) = x(t) + u(t) + w(t) \\
y(t) = x(t) + v(t)
\]

Example: open-loop stable system
\[
\dot{x}(t) = -x(t) + u(t) + w(t) \\
y(t) = x(t) + v(t)
\]
Structure of Stationary Points

Implication

**Corollary:** Consider gradient descent iterations

\[ K_{t+1} = K_t - \alpha \nabla J(K_t) \]

If the iterates converge to a minimal controller, then this minimal controller is a global optima.

More questions:

- Escaping saddle points?
- Convergence conditions?
- Convergence speed?
- Alternative model-free parameterization?
## Comparison with LQR

<table>
<thead>
<tr>
<th>Policy optimization for LQR</th>
<th>Policy optimization for LQG</th>
</tr>
</thead>
</table>
| \[
\min_K J(K) \\
\text{s.t. } K \in \mathcal{K}
\] | \[
\min_K J(K) \\
\text{s.t. } K = (A_K, B_K, C_K) \in \mathcal{C}_{\text{full}}
\] |

### Connectivity of feasible region
- **Always connected**
- **Disconnected, but at most 2 connected comp.**
- **They are almost identical to each other**
- **Non-unique, non-isolated stationary points**
- **Spurious stationary points (strict saddle, nonminimal controller)**
- **All mini. stationary points are globally optimal**

### Stationary points
- **Unique**
- **Non-unique, non-isolated stationary points**
- **Spurious stationary points (strict saddle, nonminimal controller)**
- **All mini. stationary points are globally optimal**

### Gradient Descent
- **Gradient dominance**
- **Global fast convergence (like strictly convex)**
- **No gradient dominance**
- **Local convergence/speed (unknown)**
- **Many open questions**

### References
- Fazel et al., ICML, 2018; Malik et al., 2019; Mohammadi et al., IEEE TAC, 2020; Li et al., 2019; K. Zhang, B. Hu, and T. Başar, 2021; Furieri et al., 2019; Feiran Zhao & Keyou You, 2021, and many others
- Zheng*, Tang*, Li. 2021, [link](#) (* equal contribution)
Conclusions
Policy optimization for LQG control

- Much richer and more complicated than LQR
- Disconnected, but at most 2 connected components
- Non-unique, non-isolated stationary points, strict saddle points
- Minimal (controllable and observable) stationary points are globally optimal
Ongoing and Future work

- How to certify the optimality of a non-minimal stationary point
- Perturbed policy gradient (PGD) for escaping saddle points
- Quantitative analysis of PGD algorithms for LQG
- Alternative model-free parametrization of dynamical controllers (e.g., Makdah & Pasqualetti, 2023; Zhao, Fu & You, 2022.)

✓ Better optimization landscape structures, smaller dimension

Analysis of the Optimization Landscape of Linear Quadratic Gaussian (LQG) Control

Thank you for your attention!

Q & A


Role of convex parameterization

Message: favorable landscape properties for nonconvex $J$ can be obtained \textit{from} the convex parameterization under appropriate conditions on the mapping

[Sun, F.,’21 ; Umenberger et al.’22 ; Hu et al.’23 survey]

Warm up: convex formulation for continuous-time LQR

\[
\begin{align*}
\min_{Z,L,P} & \quad \text{Tr}(QP + ZR) \\
\text{s.t.} & \quad AP + PA^T + BL + LB^T + \Sigma = 0, \\
& \quad P \succ 0, \\
& \quad \begin{bmatrix} Z & L \\ L^T & P \end{bmatrix} \succeq 0
\end{align*}
\]

\[\Rightarrow \min_{Z,L,P} f(L, P, Z)\]
\[\text{s.t.} \quad (L, P, Z) \in S\]

where $K^* = L^* (P^*)^{-1}$.

\begin{itemize}
\item further, $K = LP^{-1}$ parameterizes \textit{all} stabilizing $K \in \mathcal{K}$
\item also see [Mohammadi et al.’19]
\end{itemize}
Assumptions on parameterization map

\[
\min_K J(K) \quad \text{s.t.} \quad K \in \mathcal{K}
\Rightarrow
\min_{Z, L, P} f(L, P, Z) \quad \text{s.t.} \quad (L, P, Z) \in S
\]

Assumptions:

1. \( S \) is convex, \( f(L, P, Z) \) is convex, bounded, differentiable on \( S \).
2. we can express \( J(K) \) as

\[
J(K) = \min_{L, P, Z} f(L, P, Z), \quad \text{s.t.} \quad (L, P, Z) \in S, \quad K = LP^{-1}.
\]

more generally, \( K = LP^{-1} \) can be replaced by a surjective map \( K = \Phi(L, P) \) with “nicely behaved” first-order derivatives.

[Sun, F.,’21], [Umenberger et al.,’22]
Theorem (simplified) [Sun & F.,'21]

Under assumptions 1 and 2,

\[ \nabla J(K) = 0 \iff K = K^*. \]

Also,

- If \( f \) is convex, \( \| \nabla J(K) \|_F \gtrsim J(K) - (K^*) \).
- If \( f \) is \( \mu \)-strongly convex, \( \| \nabla J(K) \|_F \gtrsim (\mu (J(K) - J(K^*)))^{1/2} \).

\( \gtrsim \) hides instance-dependent constants; depend on system parameters & initial point \( K_0 \)
\[
\min_{K \in \mathcal{K}} J(K)
\]

\[
\min_{Z, L, P} f(L, P, Z),
\]

s.t., \( (L, P, Z) \in S \)

\[\nabla J(K_t)\]

Map the point

Map the gradient

Map the direction

\[\nabla f(L_t, P_t, Z_t)\]
A general version that applies to non-smooth $J(K)$ as well:

**Theorem [Hu et al.,’23]**

Suppose $J(K)$ is differentiable or subdifferentially regular, Assumptions 1, 2 hold. For any $K$ satisfying $J(K) > J(K^*)$, there exists non-zero $V$ in the descent cone of $\mathcal{K}$ at $K$, such that

$$0 < J(K) - J(K^*) \leq -J'(K, V),$$

so any stationary point of $J$ is a global minimum.

$J'(K, V)$ denotes directional derivative of $J(K)$ along direction $V$. When $J$ is differentiable, $J'(K, V) = Tr(V^T \nabla J(K))$. 
Example: Continuous time LQR

\[
\min_{Z,L,P} f(L, P, Z) := \text{Tr}(QP) + \text{Tr}(ZR)
\]

s.t., \( A(P) + B(L) + \Sigma = 0, \ G \succ 0, \)

\[
\begin{bmatrix}
Z & L^\top \\
L & G
\end{bmatrix} \succeq 0
\]

**Question**: \( K = LP^{-1} \), is \( P \) always invertible? (yes, if initial \( x_0 \) has full-rank covariance)

\( L, P, P^{-1} \) are bounded in the sublevel set \( \{ K : J(K) \leq a \} \).

then: \( a \geq J(K) = \text{Tr}(QP) + \text{Tr}(LP^{-1}L^\top R) \).
Example: Continuous time LQR

$L, P, P^{-1}$ are bounded in the sublevel set $\{K : (K) < a\}$.

Define

$$\nu = \frac{\lambda^2_{\min}(\Sigma)}{4} \left( \sigma_{\max}(A) \lambda^{-1/2}_{\min}(Q) + \sigma_{\max}(B) \lambda^{-1/2}_{\min}(R) \right)^{-2},$$

then

$$\|J(K)\| \leq -C_1(J(K) - J(K^*))$$

where

$$C_1 = \frac{\nu \lambda^{1/2}_{\min}(Q) \lambda^{1/2}_{\min}(R)}{4a^4} \cdot \min \left\{ a^2, \nu \lambda_{\min}(Q) \right\}.$$
Many other landscape results rely on connections to LMIs

$\mathcal{H}_\infty$ landscape: Clarke stationary is global [Guo et al., 2022]

Dynamic filtering: Differentiable convex lifting [Umenberger et al., 2022]

LQG: Connectivity [Y. Zheng, 2023]

Output-feedback $\mathcal{H}_\infty$: Connectivity [Hu et al., 2022]

A general tool for landscape study. More study is needed for output-feedback problems!
The last section of our survey article lists several directions:

- Further connections between optimization and control theory, e.g. complexity of escaping saddles for output feedback problems
- Advanced regularization for stability, robustness, and safety
- Nonlinear systems, deep RL, and perception-based control
- Multi-agent systems and decentralized control
- Integration of model-based and model-free methods
- New PO formulations from machine learning

And many more which are not listed in our article!